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IMPERFECTION SENSITIVITY AND POSTBUCKLING BEHAVIOR OF SHEAR-DEFORMABLE COMPOSITE DOUBLY-CURVED SHALLOW PANELS†

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Abstract—The static postbuckling of simply-supported single and multilayered composite double-curved shallow panels subjected to a system of in-plane compressive edge loads is studied. The effects caused by transverse shear deformation, lamination, the character of in-plane boundary conditions and transverse normal stress, are considered and a series of pertinent conclusions are outlined. Comparisons are made of the obtained results with their classical counterparts and conclusions related to their range of applicability are presented. Moreover, by incorporating the initial geometric imperfections, their influence on the load carrying capacity of curved composite panels is discussed and the peculiarities of their effect as compared to the case of flat panels are emphasized.

INTRODUCTION

Laminated composite structures are being increasingly used in the aeronautical and aerospace constructions. The employment in their construction of the new composite material systems exhibiting exotic properties, such as high anisotropy ratios, requires, for a more reliable prediction of their response, the adoption of refined structural models which are obtained by discarding the classical Love–Kirchhoff assumptions. Due to the importance of various shell structures as the load carrying members in aircraft, spacecraft, naval constructions, etc., the study of their stability behavior has received a great deal of attention. The large lists of references in the specialized monographs by Vol'mir (1967), Budiansky (1974), Brush and Almroth (1975), Esslinger and Geier (1975), Grigoliuk and Kabanov (1978), Bushnell (1985) and Leissa (1985) as well as in the survey papers by Hutchinson and Koiter (1970), Bushnell (1981), Arbocz (1981), Singer (1982), Citerley (1982) and Simitses (1986) fully attest this statement.

One of the important problems deserving special attention is the study of the postbuckling of laminated composite double-curved panels under compressive edge loads. The well-known postbuckling strength exhibited by metallic panels has permitted the design of conventional aircraft structural elements to operate within the postbuckling range. Evidently, a better understanding of their postbuckling behavior constitutes an essential requirement toward a rational employment of this strength.

However, in contrast to their metallic counterparts, the panels composed of advanced composite materials exhibit high flexibilities in transverse shear and, as a result, the transverse shear effects have to be incorporated. In addition, both the composite structures and the traditional metallic ones may exhibit some unavoidable geometric imperfections. Their presence could result in significant differences and sometimes even in drastic changes in their postbuckling behavior as compared to their perfect counterparts.

† Dedicated as a modest homage to the memory of the distinguished scientist, Dr Manuel Stein. This represents an updated and amended version of the papers presented at the Winter Annual Meeting of The American Society of Mechanical Engineers, Dallas, Texas, November 1990 and Pan American Congress of Applied Mechanics, Chile, January 1991.

The present paper addresses the problem of the compressive postbuckling of doubly-curved homogeneous laminated composite panels. The roles played by transverse shear deformation, transverse normal stress, the character of **in-plane** boundary conditions is discussed and a series of conclusions related to their effects on the associated buckling and limit loads and of their sensitivity to initial geometric imperfections is outlined. In addition to the influence of the above-mentioned effects, the one related to the sign of the Gaussian curvature of the mid-surface of curved panels is also investigated.

Throughout the paper, comparisons of the results obtained within the framework of a higher order theory with their first order and classical counterparts are performed and pertinent conclusions related to their range of applicability are outlined.

It should be mentioned that research on postbuckling behavior of **shear deformable composite curved panels** appears to be somewhat scarce, a fact which could clearly be inferred from the monographs by Esslinger and Geier (1975), Leissa (1985) and Bushnell (1985) and the survey papers by Citerley (1982) and Simitses (1986).

As a preparatory step toward the study of this problem, a short derivation of the governing equations of a geometrically nonlinear theory of shear-deformable composite curved panels will be presented. The theory is developed within the Lagrangian description and retains, in the spirit of von Kármán's small strains and moderately small rotations concept, the nonlinearities asssociated with the transverse displacement, only.

This paper represents a continuation and extension of a series of results previously obtained by Librescu (1975), Librescu and Stein (1988, 1990a, b) and Librescu and Chang (1990).

GEOMETRICALLY NONLINEAR THEORY OF SHEAR-DEFORMABLE LAMINATED COMPOSITE CURVED PANELS—BASIC ASSUMPTIONS

Consider the case of laminated composite double-curved shallow panels of uniform thickness h, symmetrically laminated with $2m+1$ ($m = 1, 2, \ldots$) elastic anisotropic layers. It is supposed that the layers are in perfect bond, implying that no slipping may occur between two contiguous layers.

The points in the three-dimensional space of the panel are referred to a set of curvilinear normal system of coordinates, x^i, where x^α ($\alpha = 1, 2$) denote the in-plane coordinates, while $x^3 = 0$ defines the reference surface (coinciding with the mid-surface of the mid-layer). The components of the metric tensor of the undeformed reference surface (denoted henceforth as σ) are:

$$a_{\alpha\beta} = \vec{a}_\alpha \cdot \vec{a}_\beta; \quad a_{\alpha 3} = \vec{a}_\alpha \cdot \vec{a}_3 = 0; \quad a_{33} = \vec{a}_3 \cdot \vec{a}_3 = 1,$$
$$a^{\alpha\beta} = \vec{a}^\alpha \cdot \vec{a}^\alpha; \quad a^{\alpha 3} = \vec{a}^\alpha \cdot \vec{a}^3 = 0; \quad a^{33} = \vec{a}^3 \cdot \vec{a}^3 = 1, \tag{1}$$

where $\vec{a}^i$ and $\vec{a}_i$ denote the contravariant and covariant base vectors of σ, respectively. The spatial metric tensor components g_{ij} of the undeformed shell space are connected with their two-dimensional counterparts $a_{\alpha\beta}$ by

$$g_{\alpha\beta} = \mu_\alpha^\lambda \mu_\beta^\omega a_{\lambda\omega}; \quad g_{\alpha 3} = g^{\alpha 3} = 0; \quad g_{33} = g^{33} = 1, \tag{2}$$

where

$$\mu_\beta^\alpha = \delta_\beta^\alpha - x^3 b_\beta^\alpha,$$

δ_β^α and b_β^α denoting the Kronecker delta and the mixed curvature tensor, respectively.

In order to reduce the three-dimensional elasticity problem to an equivalent two-dimensional one, the equations connecting the covariant derivatives of space tensors with their surface counterparts are used. Such relations, not restricted to the case of shallow surfaces, are:

$$T_{\alpha\|\beta} = \mu_\alpha^\nu(\bar{T}_{\nu|\beta} - b_{\nu\beta}\bar{T}_3); \quad T_{\alpha\|3} = \mu_\alpha^\nu \bar{T}_{\nu,3}; \quad T_{3\|\alpha} = \bar{T}_{3,\alpha} + b_\alpha^\nu \bar{T}_\nu; \quad T_{3\|3} = \bar{T}_{3,3}. \tag{3}$$

For more details concerning their deduction, the reader is referred, for example, to Naghdi (1963) and Librescu (1975).

Here, partial differential is denoted by a comma $(\ )_{,i} \equiv \partial/x_i$, while $(\ )\|_i$ and $(\ )|_\alpha$ stand for the covariant differentiations with respect to the space and surface metrics, respectively, while the shifted components are identified by an upper bar. In the above relationships (as well as in the following developments), Einstein's summation convention is implied for the repated indices, where Latin indices range from 1 to 3 while the Greek indices range from 1 to 2.

The conditions of shallowness of a shell are discussed in the monographs by Green and Zerna (1968) and Gol'denveizer (1976). Denoting by $Z(\equiv Z(x^\omega))$ the amount of deviation of the shell reference surface from a plane Π (measured normal to the plane), according to these conditions, Z is assumed to be small when compared with a maximum length of an edge of the shell or with the minimum radius of curvature of σ. For this case, the assumption

$$\max Z_{,\alpha} \ll 1, \tag{4}$$

gives rise to the result that the metric tensors associated with the system of coordinates on σ and with its projection on the plane Π are the same and, in addition, that the curvature tensor of the reference surface behaves as a constant in the differentiation operation.

From this result it may be inferred that if the projected coordinate curves on Π constitute a Cartesian orthogonal net, then the original ones on σ are also to be, on the basis of (4), a Cartesian orthogonal net. Due to the equivalence of the two metrics, it may also be concluded that the surface covariant differentiations may be done with respect to the metric associated with the plane Π and thus it is possible to change the order of the covariant differentiations (since the Riemann–Christoffel tensor associated with the plane vanishes).

Moreover, consistent with the shallow shell theory (SST) we may appropriately assume that

$$\mu_\beta^\alpha \rightarrow \delta_\beta^\alpha, \tag{5}$$

where μ_β^α, defined by eqn $(2)_4$ plays the role of shifter in the space of normal coordinates (Naghdi, 1963; Librescu, 1975). From (5) it may also be concluded that in this case

$$\mu \equiv |\mu_\beta^\alpha| = (g/a)^{1/2} \rightarrow 1, \tag{6}$$

where $g \equiv \det(g_{ij})$ and $a \equiv \det(a_{\alpha\beta})$.

DISPLACEMENT REPRESENTATION AND STRAIN MEASURES

The theory of shear-deformable shallow curved panels will be developed by using the following representation of the shifted displacement components (Reddy and Liu, 1987; Librescu *et al.*, 1989; Librescu and Stein, 1988, 1990b; Dennis and Palazotto, 1990; Simitses and Anastasiadis, 1991):

$$\bar{V}_\alpha(x^\omega, x^3) = u_\alpha + x^3\psi_\alpha + \underline{(x^3)^2\lambda_\alpha + (x^3)^3\zeta_\alpha}, \quad \bar{V}_3(x^\omega, x^3) = u_3, \tag{7}$$

where

$$u_i \equiv u_i(x^\omega); \quad \psi_\alpha \equiv \psi_\alpha(x^\omega); \quad \lambda_\alpha \equiv (x^\omega); \quad \zeta_\alpha \equiv \zeta_\alpha(x^\omega).$$

Based on the representation (7), the exact fulfillment of tangential static conditions on the bounding surfaces $x^3 = \pm h/2$ (i.e. $[s^{\alpha 3}]_{-h/2}^{h/2} = 0$ and $[x^3 s^{\alpha 3}]_{-h/2}^{h/2} = 0$), yields

$$\lambda_\alpha = 0 \quad \text{and} \quad \zeta_\alpha = -\frac{4}{3h^2}(u_{3,\alpha}+\psi_\alpha). \tag{8}$$

Although a cubic variation of in-plane displacement components through the thickness was postulated, in light of eqn (8) the displacement field contains the same unknown functions as the first order transverse shear deformation theory (FSDT), i.e. u_3, u_α and ψ_α. The underlined terms in eqn (7) have the character of corrective terms allowing one to fulfill the conditions on $x_3 = \pm h/2$. In the following developments the terms generated by these higher order terms are identified by the tracer δ_H.

We will also assume the existence of an initial out-of-plane, stress-free geometrical imperfection $\mathring{u}_3 = \mathring{u}_3(x^\omega)$. By convention, the transverse deflection is measured from the imperfect surface, in the positive, inward direction.

In conjunction with the Lagrangian description, and in the spirit of small strains and moderately small rotations approximation, the strain–displacement relationship results as:

$$2e_{ij} = V_{i\|j}+V_{j\|i}+V_{3\|i}V_{3\|j}. \tag{9}$$

This expression corresponds to the **partially geometrically nonlinear theory** and constitutes the kinematical basis of the classical von Kármán theory of plates and shallow shells as well as of their refined counterparts (Librescu, 1975). By virtue of eqns (7)–(9) and taking into account the effect of geometric imperfection, the non-zero components of the strain tensor may be expressed as:

$$e_{\alpha\beta} = \varepsilon_{\alpha\beta}+x^3\kappa_{\alpha\beta}+(x^3)^3\zeta_{\alpha\beta}, \quad e_{\alpha 3} = \varepsilon_{\alpha 3}+(x^3)^2\lambda_{\alpha 3}, \tag{10}$$

where

$$2\varepsilon_{\alpha\beta} = u_{\alpha|\beta}+u_{\beta|\alpha}-2b_{\alpha\beta}u_3+u_{3,\alpha}u_{3,\beta}+u_{3,\alpha}\mathring{u}_{3,\beta}+\mathring{u}_{3,\alpha}u_{3,\beta}, \quad 2\kappa_{\alpha\beta} = \psi_{\alpha|\beta}+\psi_{\beta|\alpha},$$

$$2\zeta_{\alpha\beta} = -\delta_H\frac{4}{3h^2}(2u_{3|\alpha\beta}+\psi_{\alpha|\beta}+\psi_{\beta|\alpha}),$$

$$2\varepsilon_{\alpha 3} = \psi_\alpha+u_{3,\alpha}+b^\rho_\alpha u_\rho, \quad 2\lambda_{\alpha 3} = -\delta_H\frac{4}{3h^2}(\psi_\alpha+u_{3,\alpha}+b^\rho_\alpha u_\rho). \tag{11}$$

It should be mentioned that: (i) the strain components associated with the FSDT may be obtained from eqns (10) by considering therein $\zeta_{\alpha\beta}$ and $\lambda_{\alpha 3}$ as zero quantities (or equivalently by considering $\delta_H = 0$), while (ii) for

$$\psi_\alpha \rightarrow -(u_{3,\alpha}+b^\rho_\alpha u_\rho), \tag{12}$$

the strain measures (11) reduce to the ones associated with the classical theory (based on the Love–Kirchhoff hypothesis).

In the previous (and forthcoming) equations the covariant differentiation is performed with respect to the metric at Π.

CONSTITUTIVE EQUATIONS

The three-dimensional constitutive equations associated with an elastic monoclinic material may be expressed as (Librescu, 1967, 1975):

$$s^{\alpha\beta} = \bar{E}^{\alpha\beta\omega\rho}e_{\omega\rho}+\delta_A\frac{E^{\alpha\beta 33}}{E^{3333}}s^{33}, \quad s^{\alpha 3} = 2E^{\alpha 3\omega 3}e_{\omega 3} \tag{13}$$

where

$$\tilde{E}^{\alpha\beta\omega\rho} = E^{\alpha\beta\omega\rho} - \frac{E^{\alpha\beta 33}E^{33\omega\rho}}{E^{3333}}. \tag{14}$$

In (13) and (14) E^{ijmn} and $\tilde{E}^{\alpha\beta\omega\rho}$ denote the tensors of elastic and modified elastic moduli, respectively; δ_A is a tracer identifying the contribution of s^{33} in the constitutive equations (and later in the governing equations), while s^{ij} denotes the second Piola–Kirchhoff stress tensor. In order to express s^{33} in terms of the basic unknowns, the third equation of motion of the nonlinear theory of three-dimensional elasticity, namely the one associated with the index $i = 3$ in

$$[s^{jr}(\delta^i_r + V^i\|_r)]\|_j = 0, \tag{15}$$

has to be used. Employment in that equation of the relationships connecting the covariant derivatives of space tensor with their surface counterparts [see eqns (3) and also the ones in Librescu (1975) and Naghdi (1963)], followed by its integration over the segment $[0, x^3)$, yields

$$s^{33} = -\int_0^{x^3} \{s^{\alpha 3}|_\alpha + b_{\rho\alpha}s^{\alpha\delta}(\delta^\rho_\delta - b^\rho_\delta u_3) + s^{3\alpha}_{;3}u_{3,\alpha} + (s^{\alpha\beta}u_{3,\beta})|_\alpha\}\, dx^3. \tag{16}$$

The employment of eqns (13) in conjunction with eqns (10), (11) and (14) in the equations expressing the stress-resultants $L^{\alpha\beta}$, $Q^{\alpha 3}$ and stress-couples $M^{\alpha\beta}$, the two-dimensional form of constitutive equations is obtained. In terms of the basic unknowns u_α, ψ_α and u_3 these equations are:

$$\begin{aligned}
L^{\alpha\beta} &= \tfrac{1}{2}F^{\alpha\beta\omega\rho}(u_{\omega|\rho} + u_{\rho|\omega} - 2b_{\omega\rho}u_3 + u_{3,\omega}u_{3,\rho} + u_{3,\omega}\mathring{u}_{3,\rho} + \mathring{u}_{3,\omega}u_{3,\rho}) \\
&\quad + \frac{4}{h^2}\,\delta_A K^{\alpha\beta\omega\rho}(u_{3,\omega}u_{3,\rho} + \mathring{u}_{3,\omega}u_{3,\rho} + \mathring{u}_{3,\rho}u_{3,\omega}), \\
Q^{\alpha 3} &= \left(R^{\alpha 3\omega 3} - \delta_H \frac{4}{h^2}\, P^{\alpha 3\omega 3}\right)(\psi_\omega + u_{3,\omega} + b^\rho_\omega u_\rho), \\
M^{\alpha\beta} &= \left(\frac{1}{2}\, H^{\alpha\beta\omega\rho} - \delta_H \frac{2}{3h^2}\, S^{\alpha\beta\omega\rho}\right)(\psi_{\omega/\rho} + \psi_{\rho/\omega}) - \frac{4}{3h^2}\, S^{\alpha\beta\omega\rho}u_{3|\omega\rho} \\
&\quad - \delta_A\left(K^{\alpha\beta\omega\rho} - \delta_H \frac{4}{3h^2}\, N^{\alpha\beta\omega\rho}\right)(\psi_{\omega/\rho} + u_{3|\omega\rho} + b^\sigma_\omega u_{\sigma/\rho}) - \delta_A Y^{\alpha\beta\lambda\pi}(u_{\lambda/\pi} + u_{\pi/\lambda}).
\end{aligned} \tag{17}$$

Throughout the paper, an index in angular brackets ⟨ ⟩ attached to a quantity denotes its affiliation to the layer of the laminate identified by that index. Excepting $Y^{\alpha\beta\lambda\pi}$ the expressions of the remaining rigidy quantities intervening in eqns(17) can be found in the paper by Librescu and Stein (1990b). The expression of $Y^{\alpha\beta\lambda\pi}$ is displayed in the Appendix.

In order to represent the governing equations in terms of the five unknown displacement quantities, five **macroscopic** equations of equilibrium are needed. These are derived by taking the appropriate moments of order zero and one of eqns (14) corresponding to $i = 1, 2$ and the moment of order zero of the same equations for $i = 3$.

Upon retaining the nonlinearities associated with the transverse deflection only, the two-dimensional equations of equilibrium assume the form:

$$L^{\alpha\beta}|_\beta = 0, \quad M^{\alpha\rho}|_\beta - Q^{\alpha 3} = 0, \quad L^{\alpha\beta}(u_{3,\beta} + \mathring{u}_{3,\beta})|_\alpha + b_{\rho\alpha}L^{\alpha\rho} + Q^{\alpha 3}|_\alpha + p_3 = 0, \tag{18}$$

where p_3 denotes the transversal load. Substitution of stress-resultants and stress-couples as expressed by eqns (17) and in (18) results in one of the possible forms of the governing equations of geometrically nonlinear shear-deformable **anisotropic** composite shallow panels. This form of the governing equations will not be displayed here.

A MIXED FORMULATION OF THE GOVERNING EQUATIONS

In what follows, the governing equations of the geometrically nonlinear theory of composite curved shallow panels will be reduced to a form which may be viewed as the generalization of the classical von Kármán–Mushtari large deflection theory of shallow shells to the case of geomerically imperfect shear-deformable laminated composite curved panels. To this end, the procedure developed in the papers by Librescu and Stein (1988, 1990a, b) will be followed.

In short, it reduces to: the conversion of two-dimensional constitutive equations (17) to the case of transversely-isotropic material layers (the surface of isotropy of the material of each layer being parallel to σ); the representation of stress resultants $L^{\alpha\beta}$ in terms of the Airy stress function $F(x^{\omega})$ resulting in the identical fulfillment of eqns $(18)_1$; consideration of the compatability equation

$$c^{\alpha\pi}c^{\beta\lambda}(\varepsilon_{\alpha\beta|\pi\lambda}+\tfrac{1}{2}u_{3|\alpha\beta}u_{3|\lambda\pi}+\tfrac{1}{2}\mathring{u}_{3|\pi\lambda}u_{3|\alpha\beta}+\tfrac{1}{2}u_{3|\pi\lambda}\mathring{u}_{3|\alpha\beta}+b_{\alpha\beta}u_{3|\pi\lambda})=0, \tag{19}$$

which has to be fulfilled, and finally in the representation of ψ^{α} in terms of a potential function $\phi(\equiv\phi(x^{\alpha}))$ as:

$$\psi^{\alpha}=-\frac{1}{S}c^{\omega\alpha}\phi_{,\omega}-\frac{D}{S}u_{3}|^{\alpha\omega}_{\omega}-\left(\frac{B+C}{S^{2}}-\delta_{A}\frac{M}{S^{2}}\right)p_{3}|^{\alpha}-(u_{3}|^{\alpha}+b^{\alpha}_{\rho}u^{\rho})$$
$$-\left(\frac{B+C}{S^{2}}-\delta_{A}\frac{M}{S^{2}}\right)c^{\sigma\rho}c^{\lambda\pi}\{F_{|\rho\pi}[(u_{3|\sigma\lambda}+\mathring{u}_{3|\sigma\lambda})+b_{\sigma\lambda}]\}|^{\alpha}. \tag{20}$$

By paralleling the procedure developed in the paper by Librescu and Stein (1990a, b), the following system of governing equations is obtained:

$$Du_{3}|^{\alpha\beta}_{\alpha\beta}-c^{\alpha\omega}c^{\beta\rho}\Big\{b_{\alpha\beta}F_{|\omega\rho}+(u_{3|\alpha\beta}+\mathring{u}_{3|\alpha\beta})F_{|\omega\rho}$$
$$-\left(\frac{B+C}{S}-\delta_{A}\frac{M}{S}\right)(b_{\alpha\beta}F_{|\omega\rho}+F_{|\omega\rho}(u_{3|\alpha\beta}+\mathring{u}_{3|\alpha\beta}))|^{\sigma}_{\sigma}\Big\}-p_{3}+\left(\frac{B+C}{S}-\delta_{A}\frac{M}{S}\right)p_{3}|^{\sigma}_{\sigma}=0,$$
$$(\bar{b}+\bar{c})F|^{\lambda\pi}_{\lambda\pi}+\tfrac{1}{2}(u_{3}|^{\rho}_{\rho}u_{3}|^{\lambda}_{\lambda}-u_{3}|^{\lambda}_{\rho}u_{3}|^{\rho}_{\lambda})+(\mathring{u}_{3}|^{\pi}_{\pi}u_{3}|^{\alpha}_{\alpha}-\mathring{u}_{3}|^{\pi}_{\alpha}u_{3}|^{\alpha}_{\pi})+(2Hu_{3}|^{\pi}_{\pi}-b^{\beta}_{\lambda}u_{3}|^{\lambda}_{\beta})$$
$$+2\delta_{A}\bar{d}(u_{3}|^{\lambda\rho}_{\lambda}u_{3}|_{\rho}+u_{3}|^{\lambda}_{\rho}u_{3}|^{\rho}_{\lambda}+\mathring{u}_{3}|^{\rho\pi}_{\pi}u_{3}|_{\rho}+u_{3}|^{\rho\pi}_{\pi}\mathring{u}_{3}|_{\rho}+2\mathring{u}_{3}|^{\rho}_{\pi}u_{3}|^{\pi}_{\rho})=0,$$
$$\phi-\frac{C}{S}\phi|^{\sigma}_{\sigma}=0. \tag{21}$$

In the preceding equations, $(\cdot)|^{\alpha}_{\alpha}$ and $(\cdot)|^{\alpha\beta}_{\alpha\beta}$ denote the two-dimensional Laplace and biharmonic operators, respectively, H denotes the mean curvature of σ $(2H\equiv b_{\alpha\beta}a^{\alpha\beta}=1/R_1+1/R_2)$. In addition, the expression of rigidity quantities B, C, D, M, S and $\bar{b}$, $\bar{c}$, $\bar{d}$ can be found in the paper by Librescu and Stein (1990b). Within this system, the equations incorporate the effects of transverse shear deformation and transverse normal stress, the geometric nonlinearities as well as the effect of initial geometric imperfections. In addition, within this formulation, the static conditions on the bounding surfaces of the panel are fulfilled.

Specialization of the obtained equations for $\delta_A=\delta_H=0$ and replacement in the rigidity quantities S and M of $G'_{\langle r\rangle}$ by $K^2G'_{\langle r\rangle}$ (where K^2 denotes a transverse shear correction factor), results in the first-order transverse shear deformation (FSDT) counterpart of the present higher-order shell theory. Further consideration of $G'_{\langle r \rangle} \to \infty$ yields the von Kármán–Mushtari equations of large deflection shallow classical shell theory. For the case of a single layered shell, $h_{\langle m+1 \rangle} \to h/2$ and $\Sigma_{r=1}(\cdot) \to 0$.

The linearized counterpart of eqns (21) constitutes a generalized version for shells of the Reissnerian theory of plates (1985) generalized in a series of works by Librescu (1975) and Librescu and Reddy (1989).

It should be mentioned that the linear equation $(21)_3$ (of a Helmholtz-type) defines the boundary layer effect. Its solution is characterized by a rapid decay when proceeding from the edges towards the interior of the shell. Although appearing uncoupled in the governing equations, the unknown function ϕ remains coupled with the other two functions, F and u_3, in the equations expressing the boundary conditions (five at each edge).

However, as was shown in the linear case (Pelech, 1975) and within the geometrically nonlinear theory of composite plates (Librescu and Stein, 1988, 1990a,b), for simply-supported boundaries, the function ϕ could be rendered decoupled in the boundary conditions to result as

$$\partial\phi/\partial n = 0, \tag{22}$$

at a boundary whose outward normal to the contour is $\vec{n} = n\vec{e}_n$. Since the governing equation for ϕ is homogeneous, the solution of eqn $(21)_3$, in conjunction with the associated BCs is identically zero. In such a case, eqn $(21)_3$ could entirely be discarded.

POSTBUCKLING OF SHALLOW DOUBLY-CURVED PANELS WITH RECTANGULAR PLANFORM

The postbuckling behavior of simply-supported composite doubly-curved panels with rectangular planform $(l_1 \times l_2)$ on Π will be analyzed. The points of σ are referred to a Cartesian orthogonal system of coordinates assumed to be parallel to the panel edges. We consider the case when the panel is subjected to a system of uniform in-plane biaxial compressive edge loads $\tilde{L}_1$ and $\tilde{L}_2$ and assume that during the postbuckling process no delamination may occur. For the problem considered here, the terms associated with the transversal load p_3 are discarded.

Depending upon the in-plane behavior at the edges, two cases are considered, namely:

(a) The edges are simply supported and freely movable (in the in-plane tangential direction). In addition, the panel is subjected to biaxial compressive edge loads.

(b) The edges are simply supported. Uniaxial edge loads are acting in the direction of the x_1-coordinate. The edges $x_1 = 0, l_1$ are considered freely movable (in the in-plane direction).

For these cases, by paralleling the procedure developed in Librescu and Stein (1988, 1990b) the BCs could be reduced to:

Case (a)

At $x_1 = 0, l_1$:

$$u_3 = 0;\quad L_{12} = 0;\quad Du_{3,11} + \left(\frac{B+C}{S} - \delta_A \frac{M}{S}\right) L_{11}(u_{3,11} + \mathring{u}_{3,11} + b_{11}) = 0;$$

$$\phi_{,1} = 0 \quad \text{and} \quad L_{11} = -\tilde{L}_1,$$

and at $x_2 = 0, l_2$:

$$u_3 = 0;\quad L_{21} = 0;\quad Du_{3,22} + \left(\frac{B+C}{S} - \delta_A \frac{M}{S}\right) L_{22}(u_{3,22} + \mathring{u}_{3,22} + b_{22}) = 0;$$

$$\phi_{,2} = 0 \quad \text{and} \quad L_{22} = -\tilde{L}_2. \tag{23}$$

Case (b)

At $x_1 = 0, l_1$:

$$u_3 = 0; \quad L_{12} = 0; \quad Du_{3,11} + \left(\frac{B+C}{S} - \delta_A \frac{M}{S}\right) L_{11}(u_{3,11} + \mathring{u}_{3,11} + b_{11}) = 0;$$

$$\phi_{,1} = 0 \quad \text{and} \quad L_{11} = -\tilde{L}_1,$$

and at $x_2 = 0, l_2$:

$$u_3 = 0; \quad u_2 = 0; \quad Du_{3,22} + \left(\frac{B+C}{S} - \delta_A \frac{M}{S}\right) L_{22}(u_{3,22} + \mathring{u}_{3,22} + b_{22}) = 0;$$

$$\phi_{,2} = 0 \quad \text{and} \quad L_{21} = 0. \tag{24}$$

Equations (23) and (24) reveal that the out-of-plane BCs are expressed in terms of u_3 and ϕ in an uncoupled form. Moreover, eqn $(21)_3$ considered in conjunction with the associated BCs $(23)_{4,9}$ and $(24)_{4,9}$ admits the trivial solution $\phi \equiv 0$.

This means that for the case of simply-supported boundaries, discarding the boundary-layer solution dos not constitute an approximation [as was thought by Grigoliuk and Chulkov (1966)], but an exact result, which, nevertheless, yields a simplification of the problem, entailing the reduction of the order of governing equations from 10 to eight and correspondingly, of the number of BCs to be fulfilled at each edge from five to four.

As usual, the imperfection is assumed to be of the same shape as the deflection mode. This statement is based on the conclusion (evidently valid when the load carrying capacity of the structure is governed by a limit point) that this selection would yield the most critical effect (Seide, 1974). It could easily be shown that the representations for u_3 and $\mathring{u}_3$:

$$\begin{Bmatrix} u_3(x_\alpha) \\ \mathring{u}_3(x_\alpha) \end{Bmatrix} = \begin{Bmatrix} f_{mn} \\ \mathring{f}_{mn} \end{Bmatrix} \sin \lambda_m x_1 \sin \mu_n x_2, \quad \lambda_m \equiv m\pi/l_1, \quad \mu_n = n\pi/l_2, \tag{25}$$

fulfill the **out-of-plane** BCs exactly. The **in-plane** BCs will also be satisfied on an average. The same is true for the out-of-plane BCs, namely $(23)_{3,8}$ and $(24)_{3,8}$, where the stretching quantities intervene coupled with the bending ones. To this end, the potential function F is represented as:

$$F(x_\alpha) = F_1(x_\alpha) - \tfrac{1}{2}((x_2)^2 \tilde{L}_1 + (x_1)^2 \tilde{L}_2). \tag{26}$$

Here $F_1 (\equiv F_1(x_\alpha))$ is a particular solution of eqn $(21)_2$ [determined in conjunction with (25)] while $\tilde{L}_1$ and $\tilde{L}_2$ denote the normal edge loads (considered positive in compression).

By fulfilling the conditions related to the function F_1 [see Librescu (1975) and Librescu and Stein (1990a, b)], $\tilde{L}_1$ and $\tilde{L}_2$ acquire the meaning of average in-plane normal edge loads.

In the case of the panel loaded in the direction of the x_1-coordinate only, the remaining edges being unloaded and immovable [that is of the type (b)], the condition for the immovable edges $x_2 = 0, l_2$ may be expressed in an average sense, which yields:

$$\int_0^{l_1}\int_0^{l_2} \{(\tilde{b}+\tilde{c})F_{,11} + \tilde{c}F_{,22} + (\delta_A \tilde{d} - \tfrac{1}{2})(u_{3,2})^2 + \delta_A \tilde{d}[(u_{3,1})^2 + 2u_{3,1}\mathring{u}_{3,1} + 2u_{3,2}\mathring{u}_{3,2}] + b_{22}u_3 - u_{3,2}\mathring{u}_{3,2}\}\, dx_1\, dx_2 = 0. \tag{27}$$

This equation, considered in conjunction with eqn (26), determines the fictitious average load $\tilde{L}_2$ for which the edges $x_2 = 0, l_2$ remain immovable.

For the present case, $F_1(x_\alpha)$ is given by:

$$F_1(x_\alpha) = A_1 \cos 2\lambda_m x_1 + A_2 \cos 2\mu_n x_2 + A_3 \cos 2\lambda_m x_1 \cos 2\mu_n x_2 + A_4 \sin \lambda_m x_1 \sin \mu_n x_2, \tag{28}$$

where

$$A_i \equiv \tilde{A}_i\,(f_{mn}^2 + 2f_{mn}f_{mn}^0), \quad i = 1,2,3, \quad \text{and} \quad A_4 \equiv \tilde{A}_4 f_{mn}, \tag{29}$$

while A_i ($i = \overline{1,4}$) are coefficients displayed in the Appendix. Galerkin's procedure applied to eqn $(21)_1$ (consisting of the substitution of u_3, $\mathring{u}_3$ and F expressed respectively by (25), (26) and (28) in eqn $(21)_1$ followed by its multiplication by $\sin\lambda_p x_1 \sin\mu_r x_2$ and integration of the obtained equation over the panel area) yields the equation governing the postbuckling behavior of laminated composite curved panels subjected to biaxial compressive edge loads:

$$\begin{aligned}
&(\tilde{L}_1\lambda_m^2 + \tilde{L}_2\mu_n^2)(f_{mn} + \mathring{f}_{mn}) + \Omega(\tilde{L}_1\lambda_m^4 + \tilde{L}_2\mu_n^4)(f_{mn} + \mathring{f}_{mn}) \\
&\quad + \Omega(\tilde{L}_1 + \tilde{L}_2)\,(f_{mn} + \mathring{f}_{mn})\mu_n^2\lambda_m^2 - \frac{16}{\lambda_m\mu_n l_1 l_2}(b_{11}\tilde{L}_1 + b_{22}\tilde{L}_2)\Delta_n^m \\
&= D(\lambda_m^2 + \mu_n^2)^2 f_{mn} + 2(f_{mn}^2 + 2f_{mn}\mathring{f}_{mn})(f_{mn} + \mathring{f}_{mn})\mu_n^2\lambda_m^2(\tilde{A}_1 + \tilde{A}_2) \\
&\quad - \frac{32}{3}\,\frac{\tilde{A}_4}{l_1 l_2}(f_{mn} + \mathring{f}_{mn})\,f_{mn}\lambda_m\mu_n\Delta_n^m + 2\Omega(f_{mn}^2 + 2f_{mn}\mathring{f}_{mn})(f_{mn} + \mathring{f}_{mn}) \\
&\quad \times [\tilde{A}_1\lambda_m^2\mu_n^4 + \tilde{A}_2\lambda_m^4\mu_n^2 + \tilde{A}_1\lambda_m^4\mu_n^2 + \tilde{A}_2\lambda_m^2\mu_n^4] \\
&\quad - \frac{4\times 16}{3}\Omega\,\frac{\tilde{A}_4}{l_1 l_2}(\lambda_m\mu_n^3 + \mu_n\lambda_m^3)\Delta_n^m f_{mn}(f_{mn} + \mathring{f}_{mn}) \\
&\quad + \frac{4b_{11}}{l_1 l_2}(f_{mn}^2 + 2f_{mn}\mathring{f}_{mn})\left[\frac{16}{9}\tilde{A}_3\frac{\mu_n}{\lambda_m} - \frac{16}{3}\tilde{A}_2\frac{\mu_n}{\lambda_m}\right]\Delta_n^m \qquad \sum_{m,n} \\
&\quad + \frac{4b_{22}}{l_1 l_2}(f_{mn}^2 + 2f_{mn}\mathring{f}_{mn})\left[\frac{16}{9}\tilde{A}_3\frac{\lambda_m}{\mu_n} - \frac{16}{3}\tilde{A}_1\frac{\lambda_m}{\mu_n}\right]\Delta_n^m \\
&\quad + \Omega\left\{\frac{16^2}{9l_1 l_2}(f_{mn}^2 + 2\mathring{f}_{mn}f_{mn})\tilde{A}_3\left[\frac{\mu_n^3}{\lambda_m}b_{11} + \frac{\lambda_m^3}{\mu_n}b_{22}\right]\Delta_n^m\right. \\
&\quad - \frac{16^2}{3l_1 l_2}(f_{mn}^2 + 2\mathring{f}_{mn}f_{mn})\left(\tilde{A}_1\frac{\lambda_m^3}{\mu_n}b_{22} + \tilde{A}_2\frac{\mu_n^3}{\lambda_m}b_{11}\right)\Delta_n^m \\
&\quad + \tilde{A}_4(\mu_n^4 b_{11} + \lambda_m^4 b_{22})f_{mn} + (b_{11} + b_{22})\left[\frac{16^2}{9l_1 l_2}\tilde{A}_3\lambda_m\mu_n\Delta_n^m(f_{mn}^2 + 2f_{mn}\mathring{f}_{mn})\right. \\
&\quad \left.\left. + \tilde{A}_4\lambda_m^2\mu_n^2 f_{mn}\right]\right\} + \tilde{A}_4 f_{mn}(b_{11}\mu_n^2 + b_{22}\lambda_m^2), \quad (m\,;n) = (\overline{1,M}\,;\overline{1,N}).
\end{aligned} \tag{30}$$

In eqn (30)

$$\Delta_n^m \equiv \Delta_{n,2r-1}^{m,2p-1} = \begin{cases} 1 & \text{if} \quad m = 2p-1 \quad \text{and} \quad n = 2r-1 \\ 0 & \text{otherwise} \end{cases}, \tag{31a}$$

the sign $\sum\limits_{m,n}$ indicates that there is no summation over the indices m and n, while

$$\Omega \equiv \frac{B+C}{S} - \delta_A\frac{M}{S}, \tag{31b}$$

denotes a reduced stiffness parameter incorporating transverse shear flexibility effect.

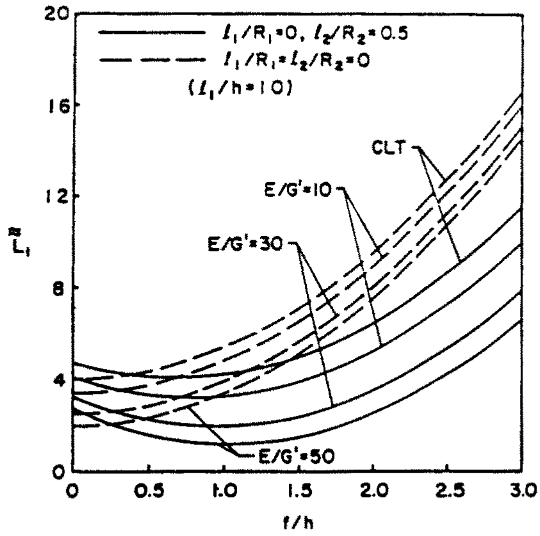

Fig. 1. Comparison of the postbuckling behavior of flat and circular cylindrical perfect panels for transverse shear-deformable and infinitely rigid in transverse shear theories. The case of free movable edges of uniaxial compression ($\tilde{L}_2 = 0$) as well as $E/E' = 5$, $\delta_A = \delta_H = 1$ were considered.

The obtained equation expressing in an explicit form the load-transversal deflection dependence is used to investigate the postbuckling behavior of curved panels. A non-dimensional form of it could be obtained by paralleling the procedure developed in the report by Nemeth (1991).

Equation (30) and its immovable counterpart could be reduced to a generic form as:

$$\tilde{L}_1/(\tilde{L}_1)_c = 1+\zeta_1\delta+\zeta_2\delta^2. \tag{32}$$

The left-hand side member of (32) expresses the ratio of the edge applied load to the buckling load of the associated perfect panel; $\delta(\equiv f/h)$ is the amplitude of deflection of the buckled mode while ζ_1 and ζ_2 are coefficients (referred to as Koiter's postbuckling parameters) which in the present case incorporate the effects of transverse shear deformation, geometry of the panel, type of loading, character of the in-plane boundary

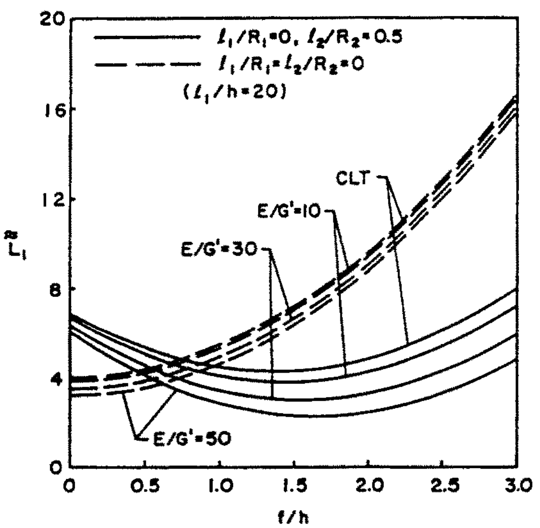

Fig. 2. Comparison of the postbuckling behavior of flat and circular cylindrical shell perfect panels for transverse shear-deformable and infinitely rigid in transverse shear theories. The case of free movable edges of uniaxial compression ($\tilde{L}_2 = 0$) as well as $E/E' = 5$, $\delta_A = \delta_H = 1$ were considered.

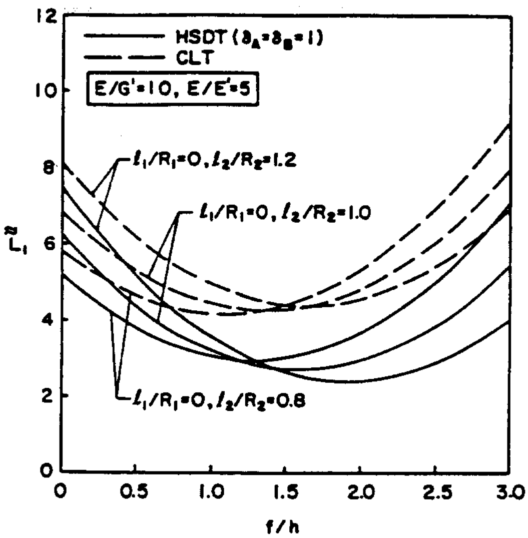

Fig. 3. Influence of the curvature ratio and transverse shear flexibility on the postbuckling behavior of geometrically perfect circular cylindrical panels with free movable edges in uniaxial compression ($\tilde{L}_2 = 0$), ($h/l = 0.1$).

conditions, etc. The sign of the coefficient ζ_1 essentially determines the imperfection sensitivity characteristics of curved panels.

Since for a flat panel (Librescu and Stein, 1988, 1990a) $\zeta_1 = 0$ and $\zeta_2 > 0$, $\tilde{L}_1/(\tilde{L}_1)_c$ results in values higher than unity, thus predicting that the perfect and imperfect flat panels can support loads in excess of the buckling load. This behavior is referred to as imperfection-insensitive. On the other hand, for shallow shells $\zeta_1 \neq 0$ and their postbuckling behavior could be sensitive or insensitive to imperfections, depending on whether $\zeta_1 < 0$ or $\zeta_1 > 0$, respectively. In the former case $\tilde{L}_1/(\tilde{L}_1)_c$ is less than unity, thus resulting in a reduction of the load-carrying capacity as compared to the perfect shell counterpart, while in the latter case, the opposite situation takes place. The equation of the type (32) was obtained and studied, e.g. in Koiter (1967a, b), Budiansky (1974), Seide (1974), Elishakoff (1980) and Birman (1990) in order to obtain information concerning the sensitivity to geometric imperfections of the load-bearing capacity of structures in the postbuckling range.

NUMERICAL ILLUSTRATIONS

The equation obtained, governing the postbuckling, determines in closed form the behavior subsequent to the onset of the buckling of laminated composite doubly-curved compressed panels. The numerical illustrations concern the cases of single and three layered composite shallow curved panels. For the latter case, two types of composite panels labelled as Case 1 and Case 2 are considered. Being the same as the ones used in the paper by Librescu and Stein (1990a, b), their characteristics will not be displayed here. It is assumed that the mid-layer of the three-layered plate is two times thicker than the external ones (implying that $h_{\langle 1 \rangle}/h$ ($\equiv h_{\langle 3 \rangle}/h$) = 0.5 and $h_{\langle 2 \rangle}/h = 0.25$). (See Fig. 16.)

It should be emphasized that in accordance with their characteristics within Case 1, the face-layers exhibit a higher flexibility in transverse shear than the core layer, while within Case 2 the opposite feature takes place. Throughout the numerical examples the case of curved and flat panels having a square projection on Π ($l_1 = l_2 \equiv l$) was considered.

Within the numerical illustrations, $\tilde{\tilde{L}}_1(\equiv \tilde{L}_1 l_1^2/\pi^2 D)$ denotes the non-dimensional uniaxial compressive load while $\delta(\equiv f/h)$ and $\delta_0(\equiv \mathring{f}/h)$ denote the non-dimensional central deflection and initial geometric imperfection of the panel, respectively. In Fig. (11), $L_R(\equiv \tilde{L}_2/\tilde{L}_1)$ defines the compressive edge load ratio, where $L_R = 0$ corresponds to the case of the uniaxial compressive load in the x_1-direction. $L_R < 0$ corresponds to the simultaneous

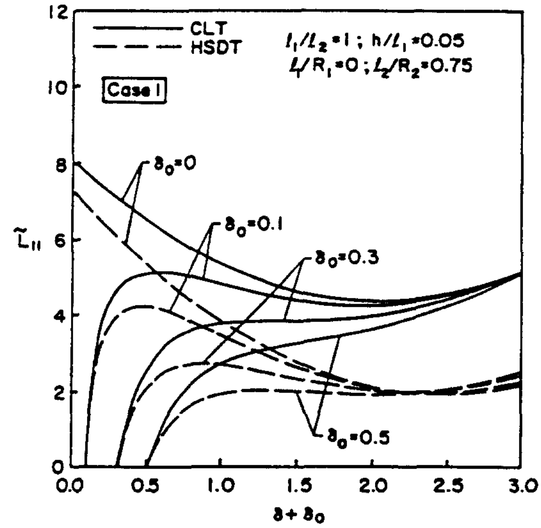

Fig. 4. The influence of geometric imperfections and transverse shear flexibility on the postbuckling behavior of three-layered circular cylindrical panels with free movable edges in uniaxial compression ($\tilde{L}_2 = 0$). (Case 1).

consideration of compressive (in the x_1-direction) and tensile (in the x_2-direction) edge loads.

CONCLUSIONS

A shear-deformable theory of geometrically nonlinear composite double-curved shallow shells symmetrically composed of transversely-isotropic material layers incorporating the effect of unavoidable geometric imperfections was developed. The associated governing equations have been recast in a form representing a generalization of some previous results obtained by Librescu (1975) and Reissner (1986) as well as of von Kármán–Mushtari classical large deflection theory of geometrically perfect shallow shells. This theory was

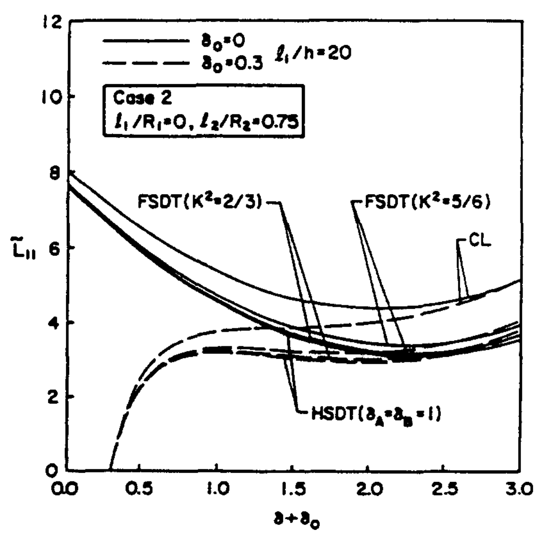

Fig. 5. Comparison of the classical FSDT and HSDT in the prediction of the postbuckling behavior of three-layered circular cylindrical panels with free movable edges in uniaxial compression ($\tilde{L}_2 = 0$). (Case 2).

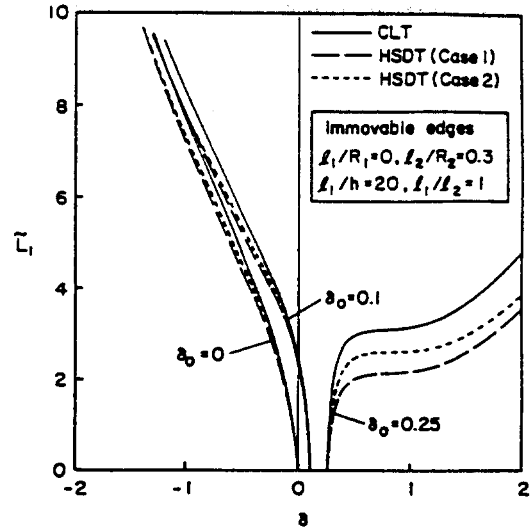

Fig. 6. The influence of initial imperfections and of transverse shear flexibility on the postbuckling behavior of three-layered circular cylindrical panels in uniaxial compression ($\tilde{L}_2 = 0$). The edges $x_1 = 0, l_2$ are considered immovable.

used to determine the postbuckling behavior of such structures. The obtained numerical results reveal that:

(a) The increase in the buckling loads of curves structures, as compared to their flat counterparts is paid by a deterioration of their behavior in the postbuckling range. In this sense, Figs 1–3 reveal that, in contrast to the flat panels which feature a **stable postbuckling behavior**, the **curved** panels are characterized by an **unstable behavior**. As the curvature of cylindrical panels diminishes, an improvement in the postbuckling behavior becomes evident (Fig. 3). This behavior is valid for both rigid in transverse shear [see Zhang and Matthews (1985) and Chia (1990)] and shear-deformable panels. Moreover, with the increase in the thickness ratio, the curved panels are more sensitive, from the postbuckling behavior point of view, to the transverse shear flexibility effect than their rigid in transverse shear counterparts (Figs 1 and 2).

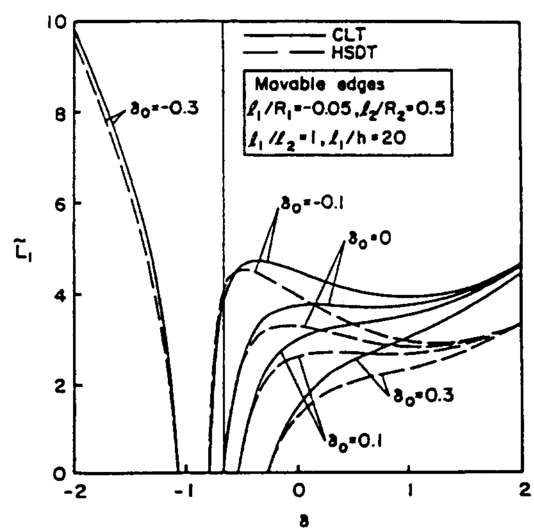

Fig. 7. The influence of positive and negative geomeric imperfections and of transverse shear flexibility on the postbuckling behavior in uniaxial compression ($\tilde{L}_2 = 0$) of doubly-curved panels of negative Gaussian curvature. The case of movable edges, $E/G' = 30$ and $E/E' = 5$ was considered.

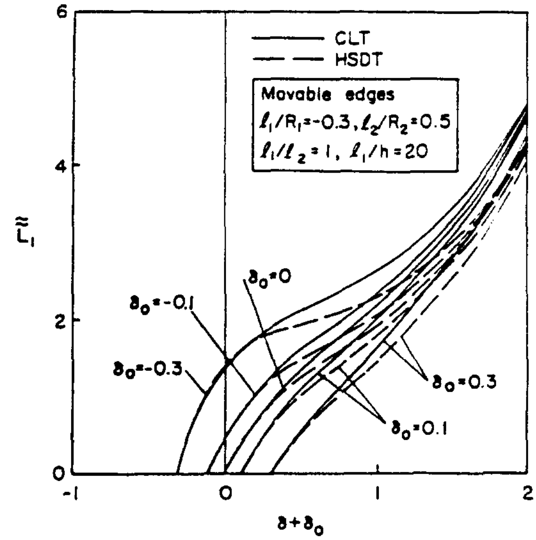

Fig. 8. The influence of geometric imperfections (positive and negative) and of transverse shear flexibility on the postbuckling behavior of doubly-curved shallow panels of negative Gaussian curvature. Within this case, $\tilde{L}_2 = 0$; $E/G' = 30$, $E/E' = 5$.

(b) In the case of circular cylindrical panels with free movable edges, in contrast to their flat counterparts [see Librescu and Stein (1990a,b)], initial positive imperfections have a deleterious effect upon their maximum load-carrying capability (Figs 4 and 5).

(c) For the circular cylindrical panels with free movable edges, the reduction in the load-carrying capability becomes more severe with the increase in the transverse shear flexibility. However, when the panel is thin, its influence on the load-carrying capability becomes almost insignificant.

(d) The geometrically **perfect** circular cylindrical panels with free movable edges subjected to compresssive loads characterized by $L_R < 0$, exhibit a postbuckling behavior similar to that of an **imperfect** panel (Fig. 11). The same figure reveals that in the case of immovable edges, the effect of the transverse shear flexibility becomes immaterial. As Fig. 6 reveals, this trend remains valid in the case of **small** geometric imperfections. However, with their increase, this trend changes. In fact, Fig. 6 shows that for $\delta_0 = 0.25$, significant

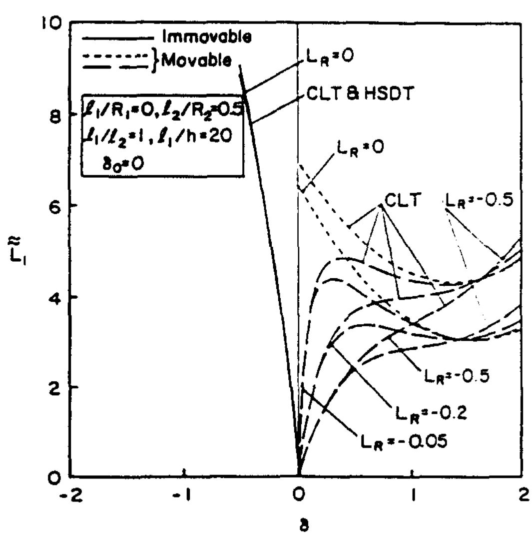

Fig. 9. The influence of bi-axial in-plane edge loads (characterized by $L_R \leq 0$) and of the character of in-plane boundary conditions (movable and immovable) on the postbuckling of perfect circular cylindrical panels. The comparison includes also CLT and HSDT. Here $E/G' = 30$, $E/E' = 0$.

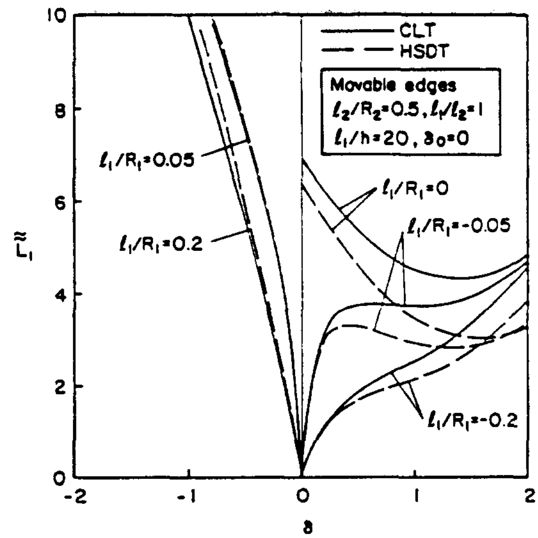

Fig. 10. The influence of the sign of the Gaussian curvature and of transverse shear flexibility on the postbuckling of perfect curved panels. Here $E/G' = 30$, $E/E' = 5$, $\tilde{L}_2 = 0$.

differences in the postbuckling behavior of three-layered circular cylindrical panels (Case 1 and Case 2) occur. In this case, the results reveal that the cylindrical panel whose core layer is more shear deformable than the face layers (Case 2) behaves better from the postbuckling behavior point of view than its opposite counterpart (Case 1). Moreover, the circular cylindrical panels whose layers are rigid in transverse shear behave better than their shear deformable counterparts (Case 1 and Case 2).

(e) The curved panels of non-zero (positive or negative) Gaussian curvature with free movable edges behave, in the postbuckling range, like their circular cylindrical counterparts, exhibiting initial geometric imperfections (Figs 7 and 8).

(f) The **negative** imperfections have a beneficial effect upon the load-carrying capacity of curved panels of zero and non-zero Gaussian curvatures (Figs 7–10).

(g) In contrast to the case of flat panels where the classical Kirchhoffean model provides results overestimating the true predictions of the carried compressive loads [see Librescu and Stein (1988, 1990a, b) and Stein *et al.* (1990)], in the case of curved panels, this trend

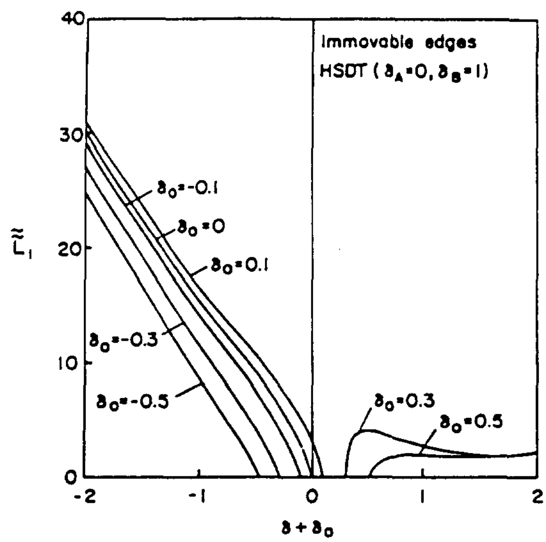

Fig. 11. Postbuckling behavior of shear deformable circular cylindrical panels with immovable edges, subjected to uniaxial compressive loads ($\tilde{L}_2 = 0$) and exhibiting positive and negative initial geometric imperfections. Here $l_1/R_1 = 0$; $l_2/R_2 = 0.5$, $h/l = 0.005$, $E/G' = 50$; $E/E' = 5$.

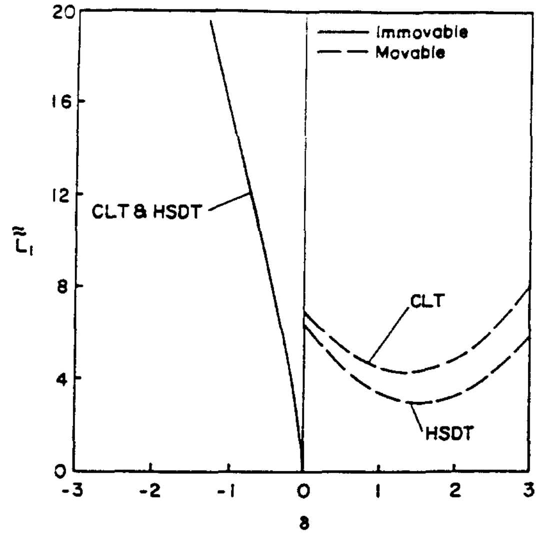

Fig. 12. The postbuckling behavior of shear deformable and rigid in transverse shear circular cylindrical perfect panels ($l/R_1 = 0$, $l/R_2 = 0.5$) having free movable and immovable edges. Here $\tilde{L}_2 = 0$; $l/h = 20$; $E/G' = 30$; $E/E' = 5$.

is not always evident. Indeed there are cases where the results are strongly affected by the sign of geometric imperfections, the sign of the Gaussian curvature, the character of in-plane boundary conditions (Figs 6, 7, 11–13) in the sense that, in many such cases, the classical and shear-deformable theories give the same (or almost the same) results in the postbuckling range.

(h) Before drawing any conclusions about the results displayed in Figs 14 and 15, it is noteworthy (see in this sense Figs 4, 6, 10, 11) that the postbuckling behavior of geometrically imperfect circular cylindrical panels could be characterized: (i) either by a limit load (denoted by $(\tilde{L}_{11})_*$ followed by a snap-through), or (ii) by a monotonous, nonlinear increase in the load-carrying capability. The curves in Figs 14 and 15 give a measure of the sensitivity of **limit loads** to the initial geometric imperfections. These figures reveal that the increase in transverse shear flexibility is accompanied by an increase in the susceptibility to the former postbuckling behavior.

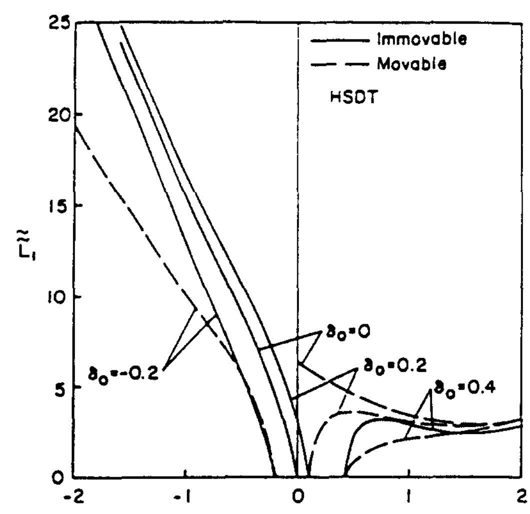

Fig. 13. The influence of the positive and negative imperfections on the postbuckling behavior of shear-deformable circular cylindrical panels having free movable and immovable boundary conditions. Here $\tilde{L}_2 = 0$; $l/R_1 = 0$; $l/R_2 = 0.5$; $l/h = 20$; $E/G' = 30$; $E/E' = 5$.

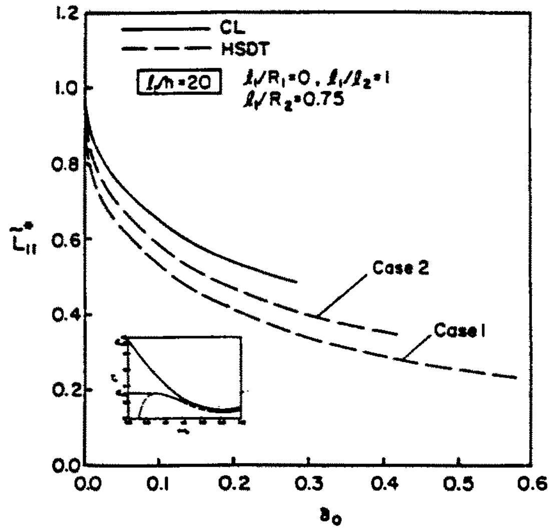

Fig. 14. Variation of the load parameter $\tilde{L}^*_{11}(\equiv(\tilde{L}_{11})_i/(\tilde{L}_{11})_c)$ with the normalized imperfection amplitude $\delta_0(\equiv f_0/h)$ for the two cases of three-layered circular cylindrical panels with free movable edges. The parts which do not appear in the graph correspond to the postbuckling behavior where the load bearing capability of the structure increases monotonously with $\delta+\delta_0$. Here $\tilde{L}_2 = 0$.

(i) For the case of single-layered circular cylindrical panels, the FSDT with $K^2 = 5/6$ provides results in excellent agreement for both the bifurcation and postbuckling responses with its HSDT counterparts. However, in the case of composite laminates, $K^2 = 2/3$ appears to be a more reliable shear correction factor than $K^2 = 5/6$. The same results were revealed to be valid also in the case of composite flat panels (Librescu and Stein, 1988, 1991).

One last remark concerns the trend occurring in the deep postbuckling range, where the curves associated with the perfect curved panels instead of becoming very close to their imperfect counterparts are intersected by them (e.g. Figs 4 and 5). A similar trend appears in the results obtained by Vol'mir (1967). As was revealed (Souza, 1990), this trend is a result of the small strains and moderately small rotations approximation [eqn (9)], used in this paper. The employment of a more general relationship, based on the small and moderate rotation approximation [developed by Librescu (1982), Librescu and Schmidt (1988) and Dennis and Palazotto (1990)], would definitely modify this trend. This statement was checked

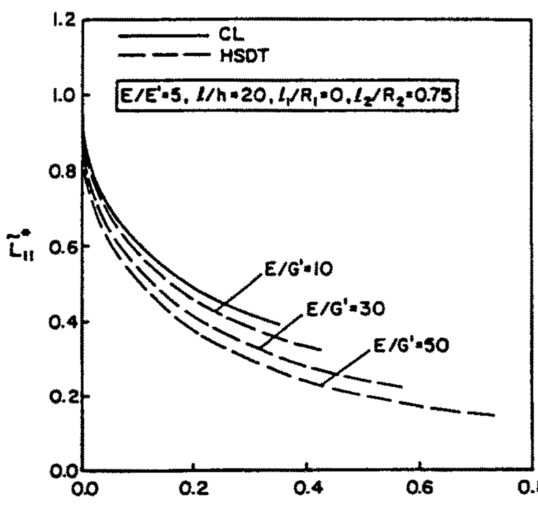

Fig. 15. Variation of the load parameter $\tilde{L}^*_{11}$ vs δ_0 for a relatively thick circular cylindrical panel. The edges are assumed to be free movable and $\tilde{L}_2 = 0$.

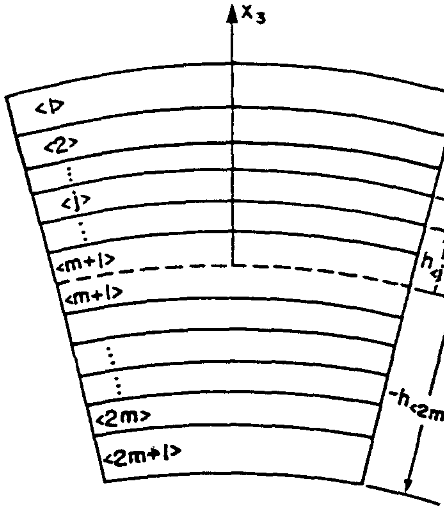

Fig. 16. Geometry of the cross-section of a composite laminated curved panel.

within a simple but comprehensive model (Souza, 1987) which succeeds in simulating the two geometrically nonlinear theories mentioned above.

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APPENDIX

The expressions of the rigidity $Y^{\alpha\beta\omega\rho}$ intervening in eqns (17):

$$Y^{\alpha\beta\omega\rho} = \tfrac{1}{4}b_{\sigma\lambda}\left\{\frac{{}_{\langle m+1\rangle}E^{\alpha\beta 33}\,{}_{\langle m+1\rangle}\bar{E}^{\sigma\lambda\omega\rho}}{{}_{\langle m+1\rangle}E^{3333}}h^2_{\langle m+1\rangle} + \sum_{r=1}^{m}\frac{{}_{\langle r\rangle}E^{\alpha\beta 33}\,{}_{\langle r\rangle}\bar{E}^{\sigma\lambda\omega\rho}}{{}_{\langle r\rangle}E^{3333}}(h^2_{\langle r\rangle} - h^2_{\langle r+1\rangle})\right\}.$$

The expressions of the coefficients $\bar{A}_i$ $(i = \overline{1,4})$ intervening in the eqns (28) and (29):

$$\bar{A}_1 = \frac{1}{16\lambda_m^4}\left\{\frac{b(b+2c)}{3(b+c)}\lambda_m^2\mu_n^2 + \delta_A\frac{4}{h^2}\frac{db}{b+c}\lambda_m^2(\mu_n^2-\lambda_m^2)\right\},$$

$$\bar{A}_2 = \frac{1}{6\mu_n^4}\left\{\frac{b(b+2c)}{2(b+c)}\lambda_m^2\mu_n^2 + \delta_A\frac{4}{h^2}\frac{db}{b+c}\mu_n^2(\lambda_m^2-\mu_n^2)\right\},$$

$$\bar{A}_3 = \delta_A\frac{4}{h^2}\frac{db}{b+c},$$

$$\bar{A}_4 = \frac{b(b+2c)}{b+c}\frac{\lambda_m^2/R_2+\mu_n^2/R_1}{(\lambda_m^2+\mu_n^2)^2},$$

where $b_{11} = 1/R_1$; $b_{22} = 1/R_2$.